daß kältere Randschichten die Messung verfälschen, kann dadurch umgangen werden, daß man mit einer sauberen, von absorbierenden Atomen freien Fenstersspülung den störenden Einfluß dieser Schichten stark reduziert. Außerdem besteht die Möglichkeit, die Bestimmung der Umkehrtemperatur in den von den Randschichten kaum beeinflußten Linienflügeln durchzuführen. Andererseits erlaubt gerade die Tatsache, daß das Plasma für die Kaliumresonanzlinien optisch dick ist und die Strahlung hier annähernd Hohlraumstrahlungsintensität bei der Elektronentemperatur erreicht, eine sehr einfache Abschätzung des Strahlungsverlustes, wenn man den spektralen Intensitätsverlauf experimentell bestimmt. Die Messungen ergaben, daß der mit Hilfe der Linienumkehrmethode ermittelte Elektronentemperaturverlauf in Abhängigkeit von der mittleren elektrischen Stromdichte in guter Übereinstimmung mit dem aus den Leitfähigkeitswerten und aus der Theorie berechneten Verlauf steht.

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Photon-Counting as a Technique for Determining Correlations in a Plasma

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It has been suggested [1] that counting photons scattered from a single-mode laser beam passing through a plasma, may be used for determining correlations in the plasma. Higher-order moments of the distribution of photon counts are related to an ensemble average of higher-order products of the electric field. Hence, this method yields also information about higher-order correlations of the electron density fluctuations, in contrast with the measurement of the scattering cross section which depends only on second-order correlations. In this paper, the connection between the statistics of photon counts and the properties of electron density fluctuations is discussed in detail, concentrating on the first and second moment of the photon count distribution. The incident wave is assumed to be plane and ideally monochromatic. For stable plasmas in a steady state, the results are written in terms of the distribution functions of the electron and ion velocities. Weakly turbulent unstable plasmas in the phase of exponential growth are also considered, using as a characteristic parameter the spectrum of the electric micro-field.

1. Introduction

Utilizing the highly monochromatic, intense light sources provided by lasers, it has become possible to obtain the spectrum of electron density fluctuations in a plasma, which is equivalent to determining the second-order density correlations in space and time, through the measurement of the cross section for light scattering [2, 3]. On the other hand, the light generated by a laser in continuous operation has also a high degree of *coherence*, owing to the constancy of its frequency and intensity (of which no use is being made in scattering experiments aiming at the determination of the cross section). This property suggests that it should also be possible to determine *higher-order correlations* of the electron density in a plasma from the scattering of laser light. One way to achieve this is by investigating the *statistics of the scattered* photons, which can be done by *photon-counting* [1]. The method is

[1] B. Crosignani and P. Di Porto, Phys. Lett. **24 A**, 69 [1967].
[2] See, e. g., G. Bekefi, Radiation Processes in Plasma, J. Wiley, New York 1966.
[3] I. B. Bernstein, S. K. Trehan, and M. P. H. Weenink, Nucl. Fusion **4**, 61 [1964].

essentially based on the fact that the distribution of photon counts, registered on the sensitive surface of a suitable detector, is determined by the correlation functions of the electromagnetic wave field (cf. Refs. [4–6]); these in turn reflect directly the correlations between the plasma electrons, provided that the incident wave is sufficiently coherent. Of course, the method is not limited to plasmas, but can be used to obtain correlations in any scattering medium. Its application to liquids has been discussed in Ref. [7].

In this paper we want to discuss in more detail the connection between the statistics of photon counts and the correlations of the electron density fluctuations in a plasma, briefly outlined in Ref. [1].

In Sect. 2 we recall the basic expressions for the moments of the photon-count distribution in terms of the correlation function of the electromagnetic field. The relation of the second-order correlation function to the conditional probability for pairs of photon counts is also touched upon. In Section 3 these correlation functions are specialized for the case of light scattered from a non-relativistic, low density plasma, considering both the case that the counter receives only the scattered light and the one that the incident wave is also collected. Explicit formulae in terms of the Fourier components of the electron density fluctuations are given for the first- and second-order correlation functions, supposing for simplicity the plasma to be spatially uniform and in a steady state. The special case of fluctuations of a frequency much lower than the frequency of the incident wave is considered in Section 4. It covers in particular all fluctuations, relevant for light scattering, which are excited by the microscopic dynamics of a non-relativistic low-density plasma in the absence of external fields. These can be evaluated, e. g., from the "microscopic Vlasov equation" in a well-known way [8–11]. As a result, the statistical properties of the scattered light can be expressed, for stable plasmas in a steady state, in terms of the electron and ion distribution functions, and for weakly turbulent, unstable plasmas in the regime of exponential growth, in terms of the spectrum of the fluctuating electric micro-field. In Section 5 the results are summarized and some experimental implications are touched upon.

2. Basic Relations

In a photon-counting experiment, one may determine the number of photons $C(t)$, recorded by a suitable detector during a given time interval $(0, t)$. This number being a random variable, its distribution function $p[C(t)]$ can be obtained by repeating the experiment many times under equivalent conditions. The dependence of this distribution function on the statistical properties of the electromagnetic field under investigation is most easily expressed with the help of its factorial moments

$$\ll \frac{C!}{(C-m)!} \gg \; \equiv \sum_{C=0}^{\infty} p[C] \frac{C!}{(C-m)!},$$

which are directly related to the correlation functions of the field.

Considering for simplicity the case of a linearly polarized field, one has explicitly [6]

$$\ll \frac{C!}{(C-m)!} \gg \; = (N s)^m \int_0^t \cdots \int_0^t G^{(m)}(t', \dots, t'_m) \prod_{j=1}^{m} \mathrm{d}t'_j. \tag{1}$$

Here N is the total number of active atoms in the counter, s its sensitivity, and $G^{(m)}$ the m-th order correlation function of the electromagnetic field with respect to time at the counter, i. e.,

$$G^{(m)}(t'_1, \dots, t'_m) = \langle E^{(-)}(\mathbf{R}, t'_1) \dots E^{(-)}(\mathbf{R}, t) \; E^{(+)}(\mathbf{R}, t'_m) \dots E^{(+)}(\mathbf{R}, t'_1) \rangle \tag{2}$$

with $\mathbf{R}$ the (mean) position of the counter, $E^{(+)}$ and $E^{(-)}$ the positive and negative frequency parts of the electric field operator, and the brackets $\langle \; \rangle$ denoting the quantum-mechanical expectation value.

[4] R. J. Glauber, Phys. Rev. **130**, 2529 [1963]; **131**, 2766 [1963].
[5] L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 [1965].
[6] R. J. Glauber, in: Quantum Optics and Electronics, Les Houches 1964, Gordon and Breach, New York 1965, p. 63; and Proc. Conf. Physics of Quantum Electronics, San Juan, Puerto Rico, 1965, McGraw-Hill, New York 1966, p. 788.
[7] M. Bertolotti, B. Crosignani, P. Di Porto and D. Sette, Phys. Rev. **157**, 146 [1967].
[8] Y. L. Klimontovich, Zh. Eksperim. Teor. Fiz. **33**, 982 [1957]; (Engl. Transl. Soviet Phys. JETP **6**, 753 [1958]).
[9] I. Fidone, S. Lafleur, and Ch. Lafleur, Report EUR-CEA-FC No. 204, Fontenay-aux-Roses (Hauts-de-Seine), France 1963.
[10] S. and Ch. Lafleur, Report CEA-R 2527, Fontenay-aux-Roses (Hauts-de-Seine), France 1964.
[11] For different approaches, see, e. g., Ref. [2, 3] and the literature cited there.

The correlation functions of Eq. (2) are a special case of the general correlation functions of the electric field,

$$G^{(m)}(x_1', x_2', \ldots, x_{2m}') = \langle E^{(-)}(x_1') \ldots E^{(-)}(x_m') \; E^{(+)}(x_{m+1}') \ldots E^{(+)}(x_{2m}') \rangle$$

where x stands for both the space and time coordinates, i. e., $x_j' = (\mathbf{r}_j', t_j')$. These describe the statistical properties of a given field completely. Let us consider in particular a situation, described by a pure state $|\alpha\rangle$, in which the positive frequency part of the electric field is uniquely determined (cf. Refs. [4, 6]), so that one has

$$\boldsymbol{E}^{(+)}(\mathbf{r}, t)|\alpha\rangle = \mathsf{E}_\alpha(\mathbf{r}, t)|\alpha\rangle$$

and $$\langle\alpha|\boldsymbol{E}^{(-)}(\mathbf{r}, t) = \langle\alpha|\mathsf{E}_\alpha^*(\mathbf{r}, t)$$

where the function $\mathsf{E}_\alpha(\mathbf{r}, t)$ is the corresponding eigenvalue of $\boldsymbol{E}^{(+)}(\mathbf{r}, t)$. Such a state may be called "fully coherent". It is immediately seen that in this case the correlation functions reduce to the factorized form

$$G^{(m)}(x_1', \ldots, x_{2m}') = \prod_{j=1}^{m} \mathsf{E}_\alpha^*(x_j') \prod_{j=m+1}^{2m} \mathsf{E}_\alpha(x_j') = \prod_{j=1}^{m} G^{(1)}(x_j', x_{2m+1-j}') .$$

In the special case $m=1$ this is, in fact, the classical condition for coherence as defined through an interference experiment [5]. Furthermore, the preceding relation, together with Eq. (1), shows that for a fully coherent state the factorial moments of the distribution of photon counts reduce to

$$\ll \frac{(C-m)!}{C!} \gg = \ll C \gg^m ,$$

that is, the distribution is Poissonian. It is important to observe that an ideally stable laser oscillating in a single mode generates just such a fully coherent electromagnetic field. For reference, it may be noted that for a Gaussian distribution of coherent states [6] the factorial moments would be given by

$$\ll \frac{C!}{(C-m)!} \gg = m! \ll C \gg^m .$$

If the state of the field may be described by a statistical superposition of fully coherent states and the contributions due to vacuum fluctuations can be neglected [4, 6], the correlation functions $G^{(m)}$ can be written

$$G^{(m)}(x_1', \ldots, x_2') = \langle \mathsf{E}_\alpha^*(x_1') \ldots \mathsf{E}_\alpha^*(x_m') \; \mathsf{E}_\alpha(x_{m+1}') \ldots \mathsf{E}_\alpha(x_{2m}') \rangle$$

where the brackets $\langle \;\rangle$ now represent simply an ensemble average. In this case a classical theory is adequate for the calculation of the amplitude of the electromagnetic field. In practice this means that one may also use, instead of $\mathsf{E}_\alpha(\mathbf{r}, t)$, the classical "analytical signal"

$$\hat{E}(\mathbf{r}, t) = \mathsf{P}_+\{E(\mathbf{r}, t)\} \equiv 2 \int_0^\infty \frac{d\omega}{2\pi} a(\mathbf{r}, \omega) \exp(i\omega t) \tag{3}$$

where $a(\mathbf{r}, \omega)$ is the Fourier transform in time of the classical amplitude of the electric field $E(\mathbf{r}, t)$,

$$a(\mathbf{r}, \omega) = \int_{-\infty}^{+\infty} dt \, E(\mathbf{r}, t) \exp(-i\omega t) . \tag{3'}$$

In the following we shall interpret the formula of Eq. (2) in this sense.

It may be noted that there is a more direct way of determining the correlation functions

$$G^{(m)}(t_1', \ldots, t_m'),$$

which has also some experimental advantages. It consists in measuring the probability of delayed coincidences of two or more photons at the counter. For example, the conditional probability $p_c(t, t+\tau) \, \Delta\tau$ that a photon count will be registered in the time interval $t+\tau \leqq t' \leqq t+\tau+\Delta\tau$, if one has occurred at time t, is given by [12]

$$p_c(t, t+\tau) = \ll \frac{dC(t)}{dt} \frac{dC(t+\tau)}{d\tau} \gg \Big/ \ll \frac{dC(t)}{dt} \gg \tag{4}$$

or explicitly [6]

$$p_c(t, t+\tau) = N s \, G^{(2)}(t, t+\tau)/G^{(1)}(t) \text{ with } \tau > 0 . \tag{4'}$$

3. Photostatistics of Light Scattered by a Plasma

Let us suppose that a plane electromagnetic wave which is fully coherent as defined in Sect. 2 and whose electric vector is given by

$$\boldsymbol{E}_1(\mathbf{r}, t) = \boldsymbol{A} \cos(\boldsymbol{k}_1 \cdot \mathbf{r} - \omega_1 t)$$

passes through a plasma by which it is scattered, and let its frequency $\omega_1 > 0$ be large enough so that the coherent response of the plasma does not affect the propagation of the wave. The analytic signal corresponding to the total electric field at the counter is respectively

$$\hat{\boldsymbol{E}}(\boldsymbol{R}, t) = \hat{\boldsymbol{E}}_1(\boldsymbol{R}, t) + \hat{\boldsymbol{E}}_2(\boldsymbol{R}, t) \tag{5}$$

[12] See, e. g., Ref. [5], p. 272.

or

$$\hat{\boldsymbol{E}}(\boldsymbol{R},t) = \hat{\boldsymbol{E}}_2(\boldsymbol{R},t) \qquad (5')$$

if the counter does or does not receive also the incident wave, with

$$\hat{\boldsymbol{E}}_1(\boldsymbol{R},t) = \boldsymbol{A}\exp[i(\omega_1 t - \boldsymbol{k}_1\cdot\boldsymbol{R})] \qquad (6)$$

and $\boldsymbol{E}_2(\boldsymbol{R},t)$ describing the scattered wave. The situations corresponding to Eqs. (5) and (5′) will be referred to from now on as Case I and Case II, respectively. Explicitly we shall use in the following[13, 14]

$$\hat{\boldsymbol{E}}_2(\boldsymbol{R},t) = \mathsf{P}_+\left\{\frac{r_0}{R}\int_V \mathrm{d}^3r\left[\left[\boldsymbol{E}_1\left(\boldsymbol{r},t-\frac{|\boldsymbol{R}-\boldsymbol{r}|}{c}\right)\times\boldsymbol{s}\right]\times\boldsymbol{s}\right] n\left(\boldsymbol{r},t-\frac{|\boldsymbol{R}-\boldsymbol{r}|}{c}\right)\right\} \qquad (6')$$

$$\approx \frac{r_0}{R}\,[(\boldsymbol{A}\times\boldsymbol{s})\times\boldsymbol{s}]\int_0^\infty \frac{\mathrm{d}\omega_2}{2\pi}\exp[i\,\omega_2(t-R/c)]\,[n(\boldsymbol{k}_2-\boldsymbol{k}_1,\omega_2-\omega_1)+n(\boldsymbol{k}_2+\boldsymbol{k}_1,\omega_2+\omega_1)]$$

with

$$\boldsymbol{k}_2 \equiv (\omega_2/c)\,\boldsymbol{s}, \quad \boldsymbol{s}\equiv\boldsymbol{R}/R, \quad \text{and} \quad r_0 = e^2/m_e c^2 \qquad (7)$$

the classical electron radius; furthermore

$$n(\boldsymbol{r},t) = \sum_j \delta(\boldsymbol{r}-\boldsymbol{r}_j(t)) \qquad (8)$$

is the electron density with $\boldsymbol{r}_j(t)$ the position of the j-th electron, so that

$$n(\boldsymbol{k},\omega) \equiv \int_V \mathrm{d}^3r\int_0^T \mathrm{d}t\,\exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,n(\boldsymbol{r},t) = \sum_j\int_0^T \mathrm{d}t\,\exp\{i[\boldsymbol{k}\cdot\boldsymbol{r}_j(t)-\omega t]\}, \qquad (8')$$

V is the volume of the scattering plasma, assumed to be large with respect to the typical wave length of the electron density fluctuations, and T the duration of the experiment, supposed long compared to characteristic fluctuation periods. Eq. (6′) is valid for $R \gg k_1^{-1}$ and $V^{1/3}$, provided that the electron velocities are non-relativistic, the excursions of the electrons induced by the wave are small compared to all other relevant lengths and multiple scattering can be neglected (which requires $|\hat{E}_2| \ll |\hat{E}_1|$, that is, in thermal equilibrium, $(r_0^2/V^{2/3})\,N_e \ll 1$ with N_e the total number of scattering electrons in the plasma). Use has also been made of the fact that, because of their small mass, the scattering is prevalently due to electrons.

Limiting ourselves for simplicity to situations where $\boldsymbol{E}_2$ is parallel to $\boldsymbol{E}_1$ (i. e., to a wave whose electric vector is perpendicular to the plane of Fig. 1, or to small scattering angles), Eqs. (2) and (5) yield, for non-vanishing scattering angles, in Case I, e. g.,

$$G^{(1)}(t_1') = |\hat{E}_1(t_1')|^2 + \langle \hat{E}_2^*(t_1')\,\hat{E}_2(t_1')\rangle \qquad (9)$$

and

$$\begin{aligned}\Delta G(t_1',t_2') &\equiv G^{(2)}(t_1',t_2') - G^{(1)}(t_1')\,G^{(1)}(t_2')\\ &= 2\,\mathrm{Re}\,\{\hat{E}_1^*(t_1')\,\hat{E}_1^*(t_2')\,\langle\hat{E}_2(t_1')\,\hat{E}_2(t_2')\rangle\\ &\quad+\hat{E}_1^*(t_1')\,\hat{E}_1(t_2')\,\langle\hat{E}_2(t_1')\,\hat{E}_2^*(t_2')\rangle\\ &\quad+\hat{E}_1^*(t_1')\,\langle\hat{E}_2(t_2')\,\hat{E}_2^*(t_1')\,\hat{E}_2(t_2')\rangle\\ &\quad+\hat{E}_1^*(t_2')\,\langle\hat{E}_2(t_2')\,\hat{E}_2^*(t_1')\hat{E}_2(t_1')\rangle\}\\ &\quad+\langle\hat{E}_2^*(t_1')\,\hat{E}_2(t_1')\,\hat{E}_2^*(t_2')\,\hat{E}_2(t_2')\rangle \qquad (9')\\ &\quad-\langle\hat{E}_2^*(t_1')\,\hat{E}_2(t_1')\rangle\langle\hat{E}_2^*(t_2')\,\hat{E}_2(t_2')\rangle\,.\end{aligned}$$

where all fields are to be taken at the position $\boldsymbol{R}$. In deriving these relations,

$$\langle\hat{E}_2(\boldsymbol{R},t)\rangle = 0 \qquad (10)$$

has been used, which is a consequence of Eq. (6′) if the fluctuations are random, that is,

$$\langle n(\boldsymbol{r},t)\rangle = N_e/V = \text{const}$$

is valid, so that

$$\langle n(\boldsymbol{k},\omega)\rangle = 0 \quad \text{for} \quad \boldsymbol{k} \neq 0\,. \qquad (11)$$

In Case II, Eqs. (9) and (9′) hold with all terms containing E_1 omitted.

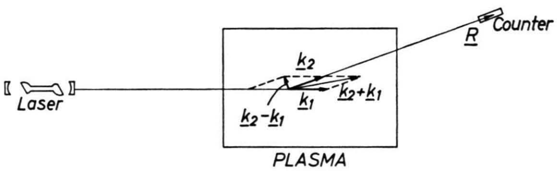

Fig. 1. Scattering of an electromagnetic wave by a plasma.

[13] The derivation of this formula is given for example in Ref. [2]; see also K. H. PANOFSKY and M. PHILLIPS, Classical Electricity and Magnetism, Addison-Wesley, London 1956.

[14] Cf. also W. H. KEGEL: Report IPP/6/21, Garching bei München, Germany 1964.

Whereas for the coherent incident wave the statistics of photon counts is Poissonian, the interaction with the plasma introduces a non-Poissonian contribution. For the second-order factorial moment this is given by

$$\Delta \equiv \langle\!\langle C^2 \rangle\!\rangle - \langle\!\langle C \rangle\!\rangle - \langle\!\langle C \rangle\!\rangle^2 = (N s)^2 \int_0^t \int_0^t \mathrm{d}t_1' \, \mathrm{d}t_2' \, \Delta G(t_1', t_2') \, ; \tag{12}$$

as Eq. (9′) shows, this is related to ensemble averages of products of the scattered electric field of second, third, and fourth order (the second and third order appearing only, if the counter receives also the incident wave [Case I]).

Putting the expressions (6) and (6′) into Eqs. (9) and (9′), the effect of the interaction with the plasma on the first and second order correlation function of the electromagnetic field may be expressed in terms of ensemble averages of products of electron density fluctuations. In general, the explicit expressions are fairly involved. Therefore we shall give them here only for the case of ensembles which are uniform in space and time. In Case I, one then obtains after some algebra

$$G^{(1)} = |A|^2 \left\{ 1 + \frac{r_0^2}{R^2} \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega_2}{2\pi} \langle |n(\boldsymbol{k}_2 - \boldsymbol{k}_1, \omega_2 - \omega_1)|^2 \rangle \right\} \tag{13}$$

and

$$G^{(2)}(t_2' - t_1') = |A|^4 \frac{r_0^2}{R^2} \lim_{T\to\infty} 2 \operatorname{Re} \int_0^{\infty} \frac{\mathrm{d}\omega_2}{2\pi} \exp[i(\omega_1 - \omega_2)(t_2' - t_1')] \tag{13′}$$

$$\cdot [\langle |n(\boldsymbol{k}_2 - \boldsymbol{k}_1, \omega_2 - \omega_1)|^2 \rangle + \langle |n(\boldsymbol{k}_2 + \boldsymbol{k}_1, \omega_2 + \omega_1)|^2 \rangle] \, ,$$

neglecting in the last expression terms of order $(r_0^4/R^4)\, N_e^2$ [those formally of order $(r_0^3/R^3)\, N_e^{3/2}$ are exactly zero for spatially homogeneaus ensembles] [15]. In Case II one has instead

$$G^{(1)} = |A|^2 \frac{r_0^2}{R^2} \lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega_2}{2\pi} \langle |n(\boldsymbol{k}_2 - \boldsymbol{k}_1, \omega_2 - \omega_1)|^2 \rangle \tag{14}$$

and

$$\Delta G(t_2' - t_1') + G^{(1)^2} \equiv G^{(2)}(t_2' - t_1')$$

$$= |A|^4 \frac{r_0^4}{R^4} \lim_{T\to\infty} \frac{1}{T} \frac{1}{(2\pi)^3} \int_0^{\infty} \mathrm{d}\omega_2 \, \mathrm{d}\omega_2' \, \mathrm{d}\omega_2'' \, \mathrm{d}\omega_2''' \cdot \delta(\omega_2 - \omega_2' + \omega_2'' - \omega_2''') \, e^{i(\omega_2 - \omega_2')(t_2' - t_1')} \tag{14′}$$

$$\begin{aligned}
\{ & \langle n(-\boldsymbol{k}_2 - \boldsymbol{k}_1, -\omega_2 - \omega_1)\, n(\boldsymbol{k}_2' + \boldsymbol{k}_1, \omega_2' + \omega_1)\, n(-\boldsymbol{k}_2'' - \boldsymbol{k}_1, -\omega_2'' - \omega_1)\, n(\boldsymbol{k}_2''' + \boldsymbol{k}_1, \omega_2''' + \omega_1) \rangle \\
+ & \langle n(-\boldsymbol{k}_2 - \boldsymbol{k}_1, -\omega_2 - \omega_1)\, n(\boldsymbol{k}_2' + \boldsymbol{k}_1, \omega_2' + \omega_1)\, n(-\boldsymbol{k}_2'' + \boldsymbol{k}_1, -\omega_2'' + \omega_1)\, n(\boldsymbol{k}_2''' - \boldsymbol{k}_1, \omega_2''' - \omega_1) \rangle \\
+ & \langle n(-\boldsymbol{k}_2 - \boldsymbol{k}_1, -\omega_2 - \omega_1)\, n(\boldsymbol{k}_2' - \boldsymbol{k}_1, \omega_2' - \omega_1)\, n(-\boldsymbol{k}_2'' + \boldsymbol{k}_1, -\omega_2'' + \omega_1)\, n(\boldsymbol{k}_2''' + \boldsymbol{k}_1, \omega_2''' + \omega_1) \rangle \\
+ & \langle n(-\boldsymbol{k}_2 + \boldsymbol{k}_1, -\omega_2 + \omega_1)\, n(\boldsymbol{k}_2' + \boldsymbol{k}_1, \omega_2' + \omega_1)\, n(-\boldsymbol{k}_2'' - \boldsymbol{k}_1, -\omega_2'' - \omega_1)\, n(\boldsymbol{k}_2''' - \boldsymbol{k}_1, \omega_2''' - \omega_1) \rangle \\
+ & \langle n(-\boldsymbol{k}_2 + \boldsymbol{k}_1, -\omega_2 + \omega_1)\, n(\boldsymbol{k}_2' - \boldsymbol{k}_1, \omega_2' - \omega_1)\, n(-\boldsymbol{k}_2'' - \boldsymbol{k}_1, -\omega_2'' - \omega_1)\, n(\boldsymbol{k}_2''' + \boldsymbol{k}_1, \omega_2''' + \omega_1) \rangle \\
+ & \langle n(-\boldsymbol{k}_2 + \boldsymbol{k}_1, -\omega_2 + \omega_1)\, n(\boldsymbol{k}_2' - \boldsymbol{k}_1, \omega_2' - \omega_1)\, n(-\boldsymbol{k}_2'' + \boldsymbol{k}_1, -\omega_2'' + \omega_1)\, n(\boldsymbol{k}_2''' - \boldsymbol{k}_1, \omega_2''' - \omega_1) \rangle \}
\end{aligned}$$

with $\boldsymbol{k}_2^{(n)} \equiv (\omega_2^{(n)}/c)\, \boldsymbol{s}$. The preceding relations can be deduced most easily by using Eq. (8′) and observing that, for spatially uniform ensembles, averages of the form $\langle \exp(i\,\boldsymbol{k} \cdot \boldsymbol{q}(t)) \rangle$ with $\boldsymbol{q}(t)$ a linear combination of electron positions $\boldsymbol{r}_j(t)$ are non-vanishing only if the latter can be combined in pairs of differences. Note also that Eq. (8′) implies the property $n(-\boldsymbol{k}, -\omega) = n^*(\boldsymbol{k}, \omega)$.

The above results show that if the counter receives also the incident wave (Case I), only infor-

[15] The expression for Δ following from Eqs. (12) and (13′) is different from the one of Eq. (3) of Ref. [1]. The latter has been based on an expression for the scattered field which is not consistent for arbitrary frequencies of the density fluctuations. It may be used, however, if these are typically small compared to the frequency ω_1 of the incident wave (cf. Section 4). In most of the practical applications this condition is satisfied.

mation about the second order (i. e., two-particle) correlations can be acquired. In particular, $\langle\langle C \rangle\rangle$ yields the spectrum of the electron density fluctuations, integrated over frequency, and $\langle\langle C^2 \rangle\rangle$ gives an expression which is related to the cosine transform (in frequency) of this spectrum (cf. Sect. 4). Higher moments are more complex functionals of the second-order correlations between electrons (to lowest significant order in $(r_0^2/R^2)\, N_e$). On the other hand, if the counter collects the scattered wave only (Case II), one obtains also knowledge about higher-order correlations. Whereas $\langle\langle C \rangle\rangle$ gives again the integrated spectrum of electron density fluctuations, $\langle\langle C^2 \rangle\rangle$ is now related to the fourth-order (i. e., four-particle) correlation function of the electrons. Generally, the moment $\langle\langle C^m \rangle\rangle$ contains information about correlations of order $2\, m$ between electrons.

4. Effect of Self-Excited Fluctuations in the Plasma

The results of the preceding Section may be appreciably simplified if the relevant fluctuation frequencies are small compared to the frequency ω_1 of the incident wave. This is usually true, e. g., for all fluctuations due to the collective interaction between plasma particles. These have namely frequencies in the range of the plasma frequencies of the particle species constituting the plasma, which must be small with respect to ω_1 if the incident wave is to propagate unaffectedly by the presence of the plasma. More generally, all relevant self-excited fluctuations in a non-relativistic low-density plasma are then of this type, since the ones associated with the free-particle motion have frequencies not larger than $k_1 \bar{\boldsymbol{v}} = \omega_1 \bar{\boldsymbol{v}}/c \ll \omega_1$, with $\bar{\boldsymbol{v}}$ a characteristic electron velocity [cf. Eq. (24)].

In such a situation $n(\boldsymbol{k}, \omega)$ is large only for $|\omega| \ll \omega_1$ as can be seen from Eq. (8′). Hence, without any significant error, Eq. (6′) with $\boldsymbol{A} \perp \boldsymbol{s}$ can be replaced by

$$\hat{E}_2(\boldsymbol{R}, t) = \frac{r_0}{R} A \int_{-\infty}^{+\infty} \frac{d\omega_2}{2\pi} e^{i\omega_2(t-R/c)}\, n\left(\frac{\omega_1}{c}\boldsymbol{s} - \boldsymbol{k}_1, \omega_2 - \omega_1\right) = \frac{r_0}{R} A\, e^{i\omega_1(t-R/c)}\, n\left(\frac{\omega_1}{c}\boldsymbol{s} - \boldsymbol{k}_1, t - R/c\right) \quad (15)$$

where

$$n(\boldsymbol{k}, t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{i\omega t}\, n(\boldsymbol{k}, \omega) = \sum_j \exp\{i\, \boldsymbol{k} \cdot \boldsymbol{r}_j(t)\}\,.$$

With

$$\mathbf{K} \equiv (\omega_1/c)\, \boldsymbol{s} - \boldsymbol{k}_1 \quad (16)$$

this yields for Case I

$$G^{(1)}(t_1') = |A|^2 \{1 + (r_0^2/R^2) \langle |n(\mathbf{K}, t_1' - R/c)|^2 \rangle\} \quad (17)$$

and, to lowest significant order,

$$\Delta G(t_1', t_2') = |A|^4 (r_0^2/R^2) \cdot 2\, \mathrm{Re}\{\langle n(\mathbf{K}, t_1' - R/c)\, n(-\mathbf{K}, t_2' - R/c) \rangle\}; \quad (17')$$

for Case II one obtains

$$G^{(1)}(t_1') = |A|^2 (r_0^2/R^2) \langle |n(\mathbf{K}, t_1' - R/c)|^2 \rangle \quad (18)$$

and

$$\Delta G(t_1', t_2') = |A|^4 \frac{r_0^4}{R^4} \{\langle |n(\mathbf{K}, t_1' - R/c)|^2 |n(\mathbf{K}, t_2' - R/c)|^2 \rangle - \langle |n(\mathbf{K}, t_1' - R/c)|^2 \rangle \langle |n(\mathbf{K}, t_2' - R/c)|^2 \rangle\}\,. \quad (18')$$

The preceding formulae are valid if the ensemble satisfies condition (11); the uniformity of the ensemble in space and time is not required. In a steady state $G^{(1)}$ is time-independent and ΔG depends only on $t_2' - t_1'$; then ΔG as given by Eq. (17′) is proportional to the cosine transform in frequency of the spectrum of fluctuations of the electron density. For the higher $G^{(m)}$ analogous expressions can be derived; in Case II one obtains in particular

$$G^{(m)}(t_1', \dots, t_m') = |A|^{2m} (r_0/R)^{2m} \langle |n(\mathbf{K}, t_1' - R/c)|^2 \dots |n(\mathbf{K}, t_m' - R/c)|^2 \rangle\,. \quad (18'')$$

For reference, we shall give in the following explicit expressions for the averages appearing on the right-hand sides of Eqs. (17) to (18′) for an isolated stable plasma in a steady state. This can be done along the lines well-known in the theory of light scattering [2, 3, 9, 14]. For unstable plasmas these averages will be written in terms of the fluctuating electric microfield, assuming a regime of exponential growth.

4a) Stable Plasma

The electron density fluctuations in a plasma can be evaluated, e. g., starting from the "microscopic Vlasov equation" [8–10], which is equivalent to the dynamical equations of motion of all the plasma particles. For a plasma consisting of electrons and one species of singly ionized ions, one has in the absence of external fields

$$\frac{\partial \nu^{(a)}}{\partial t} + \boldsymbol{v} \cdot \frac{\partial \nu^{(a)}}{\partial \boldsymbol{r}} + \frac{e_a}{m_a} \boldsymbol{E} \cdot \frac{\partial \nu^{(a)}}{\partial \boldsymbol{v}} = 0 \qquad (19)$$

where $\boldsymbol{E}(\boldsymbol{r}, t)$ is now the internal microscopic field, determined by the charges of the plasma particles through Poisson's equation

$$\operatorname{div} \boldsymbol{E} = 4 \pi e \int d^3v \, (\nu^{(i)} - \nu^{(e)}), \qquad (20)$$

$\nu^{(a)}$ being defined by

$$\nu^{(a)}(\boldsymbol{r}, \boldsymbol{v}, t) \equiv \sum_j \delta(\boldsymbol{r} - \boldsymbol{r}_j^{(a)}(t)) \, \delta(\boldsymbol{v} - \boldsymbol{v}_j^{(a)}(t))$$

and the superscript a = e, i referring to the electrons and ions, respectively; $\boldsymbol{r}_j^{(a)}(t)$ and $\boldsymbol{v}_j^{(a)}(t)$ describe the position and velocity of the j-th particle of the species a. In particular, one has

$$\int d^3v \, \nu^{(e)}(\boldsymbol{r}, \boldsymbol{v}, t) = n(\boldsymbol{r}, t).$$

Provided that the relevant fluctuations are weak and unaffected by external fields, and the plasma is homogeneous (i. e., can be described by an ensemble uniform in space) as well as on the average neutral, Eq. (19) may be replaced by

$$\frac{\partial}{\partial t} \delta\nu^{(a)} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{r}} \delta\nu^{(a)} + \frac{e_a}{m_a} \boldsymbol{E} \cdot \frac{\partial f^{(a)}}{\partial \boldsymbol{v}} = 0 \qquad (21)$$

with $f^{(a)}(\boldsymbol{v}, t) \equiv \langle \nu^{(a)} \rangle$ the one-particle distribution functions of the species a and

$$\delta\nu^{(a)}(\boldsymbol{r}, \boldsymbol{v}, t) \equiv \nu^{(a)}(\boldsymbol{r}, \boldsymbol{v}, t) - f^{(a)}(\boldsymbol{v}, t).$$

In a steady state, $f^{(a)}$ is a function of $\boldsymbol{v}$ only. It may be noted that the transition from Eq. (19) to Eq. (21) implies a "linearization" in the fluctuations $\delta\nu^{(a)}$ [9, 10].

From Eqs. (20) and (21) the Fourier components of the fluctuations of the electron density can readily be calculated (for details, see, e. g., Ref. [9]). One obtains for $\boldsymbol{k} \neq 0$

$$n(\boldsymbol{k}, \omega) = B_e(\boldsymbol{k}, \omega) \, n_L^{(e)}(\boldsymbol{k}, \omega) + B_i(\boldsymbol{k}, \omega) \, n_L^{(i)}(\boldsymbol{k}, \omega) \qquad (22)$$

where

$$B_e(\boldsymbol{k}, \omega) = \frac{1 - G_i(\boldsymbol{k}, \omega)}{1 - G_e(\boldsymbol{k}, \omega) - G_i(\boldsymbol{k}, \omega)}, \quad B_i(\boldsymbol{k}, \omega) = \frac{-G_e(\boldsymbol{k}, \omega)}{1 - G_e(\boldsymbol{k}, \omega) - G_i(\boldsymbol{k}, \omega)}, \qquad (23)$$

with

$$G_a(\boldsymbol{k}, \omega) = \frac{4 \pi e^2}{m_a k^2} \lim_{\varepsilon \to 0} \int d^3v \frac{\boldsymbol{k} \cdot \partial f^{(a)} / \partial \boldsymbol{v}}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega + i \varepsilon}; \qquad (23')$$

the quantities $n_L^{(a)}$ describe the evolution of the initial density fluctuations of electrons and ions with particle interactions neglected, that is, one has

$$n_L^{(a)}(\boldsymbol{r}, t) = \sum_j \delta(\boldsymbol{r} - \boldsymbol{r}_j^{(a)}(0) - \boldsymbol{v}_j^{(a)}(0) \, t)$$

or after Fourier transforming

$$n_L^{(a)}(\boldsymbol{k}, \omega) = 2 \pi \sum_j \exp\{i \boldsymbol{k} \cdot \boldsymbol{r}_j^{(a)}(0)\} \cdot \delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v}_j^{(a)}(0)). \qquad (24)$$

Inverting the Fourier transform in time and observing that $n_L^{(e)}$ and $n_L^{(i)}$ are statistically independent, so that

$$\langle n_L^{(e)}(\boldsymbol{k}, \omega) \, n_L^{(i)}(-\boldsymbol{k}, \omega') \rangle = 0,$$

one obtains immediately from the preceding relations

$$\langle n(\boldsymbol{k}, t) \, n(-\boldsymbol{k}, t') \rangle = V \int d^3v \exp\{-i \boldsymbol{k} \cdot \boldsymbol{v} (t' - t)\} \{ |B_e(\boldsymbol{k}, \boldsymbol{k} \cdot \boldsymbol{v})|^2 f^{(e)}(\boldsymbol{v}) + |B_i(\boldsymbol{k}, \boldsymbol{k} \cdot \boldsymbol{v})|^2 f^{(i)}(\boldsymbol{v}) \}, \qquad (25)$$

since in a steady state the average $\langle \; \rangle$ over different initial conditions can be taken using the particle distribution functions $f^{(a)}(\boldsymbol{v})$. In particular, one finds

$$\langle |n(\boldsymbol{k}, t)|^2 \rangle = V \int d^3v \{ |B_e(\boldsymbol{k}, \boldsymbol{k} \cdot \boldsymbol{v})|^2 f^{(e)}(\boldsymbol{v}) + |B_i(\boldsymbol{k}, \boldsymbol{k} \cdot \boldsymbol{v})|^2 f^{(i)}(\boldsymbol{v}) \}. \qquad (26)$$

independent of time. Furthermore, one derives

$$\begin{aligned} \langle |n(\boldsymbol{k}, t)|^2 \, |n(\boldsymbol{k}, t')|^2 \rangle = {} & V \int d^3v \, |B_e(\boldsymbol{k}, \boldsymbol{k} \cdot \boldsymbol{v})|^4 f^{(e)}(\boldsymbol{v}) + V^2 \int d^3v \, d^3v' \{1 + \exp[-i \boldsymbol{k} \cdot (\boldsymbol{v} - \boldsymbol{v}')(t' - t)]\} \\ & \cdot |B_e(\boldsymbol{k}, \boldsymbol{k} \cdot \boldsymbol{v})|^2 f^{(e)}(\boldsymbol{v}) \left\{ \frac{N_e(N_e - 1)}{N_e^2} |B_e(\boldsymbol{k}, \boldsymbol{k} \cdot \boldsymbol{v}')|^2 f^{(e)}(\boldsymbol{v}') + |B_i(\boldsymbol{k}, \boldsymbol{k} \cdot \boldsymbol{v}')|^2 f^{(i)}(\boldsymbol{v}) \right\} \\ & + \{\text{the same terms with } i \longleftrightarrow e\}. \end{aligned} \qquad (27)$$

Neglecting the terms linear in $N_e \gg 1$ with respect to those proportional to N_e^2, this reduces to

$$\langle |n(\boldsymbol{k}, t)|^2 \, |n(\boldsymbol{k}, t')|^2 \rangle = [\langle |n(\boldsymbol{k}, t)|^2 \rangle]^2 + |\langle n(\boldsymbol{k}, t) \, n(-\boldsymbol{k}, t') \rangle|^2 \qquad (27')$$

showing that in the case of a stable plasma in a steady state with weak excitation of fluctuations the fourth-order correlation function of the electromagnetic field is anyway determined by second-order particle correlations. As a result, formula (18′) may be re-written in this case as

$$\Delta G(t_1', t_2') = \Delta G(t_2' - t_1') = |A|^4 \frac{r_0^4}{R^4} |\langle n(\mathbf{K}, 0)\, n(-\mathbf{K}, t_2' - t_1') \rangle|^2 . \tag{28}$$

For times t small compared with a typical correlation time of the electron density fluctuations one hence obtains from Eqs. (1), (12) and (18)

$$\Delta \equiv \langle\!\langle C \rangle\!\rangle^2$$

which is just the relation valid for a Gaussian distribution of coherent states (cf. Sect. 2).

4b) Unstable Plasma

For an unstable plasma the calculation of the fluctuations is in general a formidable task, because the growth of unstable perturbations is limited by complex non-linear phenomena such as coupling between different wave modes and effects of the waves on the average particle distribution. Especially, the spectrum of fluctuations reached in a steady state depends strongly on the particular circumstances. The only thing that can be done relatively easy in some generality is to write, for a weakly turbulent plasma during the period of wave growth, the averages appearing in Eqs. (17) to (18′) in terms of the fluctuating electric microfield.

In this case Eq. (21) remains valid and yields, with $f^{(e)}(\boldsymbol{v}, t)$ slowly time-dependent and assuming

$$E(\boldsymbol{k}, t) \sim \exp[i \int^t \omega(\boldsymbol{k}, t')\, dt'] \tag{29}$$

where $\operatorname{Im} \omega(\boldsymbol{k}, t') < 0$, and

$$|E(\boldsymbol{k}, t)| \gg |E(\boldsymbol{k}, 0)|,$$

i. e., after some e-folding times of the instability, for the electron density fluctuations

$$n(\boldsymbol{k}, t) = \frac{i\, G_e(\boldsymbol{k}, \omega(\boldsymbol{k}, t))}{4 \pi e} \boldsymbol{k} \cdot \mathbf{E}(\boldsymbol{k}, t). \tag{30}$$

Relation (29) holds in the linear and quasi-linear regime, whenever the most unstable solution $\omega(\boldsymbol{k}, t)$ of the dispersion relation dominates the wave growth. Note that $\omega(-\boldsymbol{k}, t) = -\omega^*(\boldsymbol{k}, t)$ is valid.

Defining

$$\mathcal{E}(\boldsymbol{k}, t, t') \equiv \frac{E(\boldsymbol{k}, t)\, E^*(\boldsymbol{k}, t')}{(2\pi)^3\, V\, 8\pi} \tag{31}$$

so that

$$\mathcal{E}(\boldsymbol{k}, t, t) \equiv \mathcal{E}(\boldsymbol{k}, t) \tag{31'}$$

is the fluctuating electrostatic energy density per mode $\boldsymbol{k}$, one obtains from Eq. (30)

$$\langle n(\boldsymbol{k}, t)\, n(-\boldsymbol{k}, t') \rangle = \frac{4\pi^2}{e^2} V k^2 |G_e(\boldsymbol{k}, \omega(\boldsymbol{k}, t))|^2 \langle \mathcal{E}(\boldsymbol{k}, t, t') \rangle, \tag{32}$$

$$\langle |n(\boldsymbol{k}, t)|^2 \rangle = \frac{4\pi^2}{e^2} V k^2 |G_e(\boldsymbol{k}, \omega(\boldsymbol{k}, t))|^2 \langle \mathcal{E}(\boldsymbol{k}, t) \rangle, \tag{33}$$

and

$$\langle |n(\boldsymbol{k}, t)|^2\, |n(\boldsymbol{k}, t')|^2 \rangle = \left(\frac{4\pi^2}{e^2}\right)^2 V^2 k^4 |G_e(\boldsymbol{k}, \omega(\boldsymbol{k}, t))|^2 |G_e(\boldsymbol{k}, \omega(\boldsymbol{k}, t))|^2 \langle \mathcal{E}(\boldsymbol{k}, t)\, \mathcal{E}(\boldsymbol{k}, t') \rangle. \tag{34}$$

5. Conclusions

The preceding discussion shows that photon-counting is a useful device for gaining information about correlations in a plasma. In principle, *correlations* in electron density *of any order* can be investigated in this way. In this respect, the method is hence superior to determining the scattering cross section, which yields only the (second-order) spectrum of fluctuations, i. e., the two-particle correlations in space and time. In particular, as regards two-particle correlations in time, the difference between photon-counting and cross-section measurements is that the former yields information *directly*, while the latter require Fourier-transforming of the frequency spectrum of fluctuations primarily obtained.

As a general rule, the m-th moment of the distribution of photon counts in the scattered light is related to correlations of order $2m$ (i. e., to the correlation function of $2m$ electrons). In particular, the average number of counts in a given time interval yields the (second-order) spectrum of fluctuations integrated over frequency (i. e., the spatial correlations of two electrons). Two-particle correlations in time can also be investigated if the counter receives the scattered and the incident light simultaneously. In the case of a stable plasma in a steady state, this can be done also using the second moment of the distribution of photon counts in the scattered light alone, since here the relevant fourth-order correlations can be expressed in terms of second-order correlations in space and time. The information contained in the moments of the photon-count distribution can also be deduced from measurements of conditional probabilities.

Up to which order correlations can be obtained in practice, depends on the accuracy of the determination of the distribution function of the photon counts (for this cf. Ref. [16]) and on the relative importance of "noise photons", not produced by the scattering process. It is well-known that even for moderately high densities of the order of 10^{14} particles per cm^3 bremsstrahlung will dominate scattering completely, unless the intensity of the incident laser beam is of the order of a megawatt per cm^3. However, it is difficult to ensure the necessary coherence of the incident beam for such intensities. On the other hand, measurements of the photon-count distribution are feasible at fairly low intensities, since their statistical character allows to spread them over a rather long time interval. Hence the most suitable densities for performing such measurements should be the range of $10^{11}-10^{12}$ particles per cm^3 where cross section measurements are extremely difficult. This is true also for delayed-coincidence measurements [17], for which the problems occurring are quite similar.

A high precision of the measurement of the photon-count distribution is in particular required if second-order correlations in time are investigated by collecting the scattered and the incident light simultaneously. E. g. in thermal equilibrium, the relevant non-Poissonian term is of order $(r_0^2/R^2)\,N_e$ which for $N_e=10^{12}$ particles is typically about 10^{-16}. Hence such a measurement seems feasible only if the fluctuations contributing to the scattering are strongly enhanced with respect to the thermal level so that the non-Poissonian contribution is appreciably increased, that is in situations which are unstable or "almost" unstable (i. e., close to marginal stability [18]). The conditions may, however, be improved, too, by decreasing the intensity of the incident beam which is brought to the counter.

It must also be emphasized that the determination of time correlation through photon-counting requires an explicit time resolution of the measurements which is at least smaller than the characteristic correlation time, the latter being typically of the order of the characteristic damping or amplification times of the fluctuations. If the incident light is also collected the relevant non-Poissonian contribution to $G^{(2)}$ is oscillatory with a period of the order of the typical fluctuation periods and, consequently, the time resolution must then be of this order, too. Since all characteristic times are larger at lower densities, the low density range is more favourable for photon-counting experiments also from this point of view.

[16] F. T. Arecchi, A. Berné, A. Sona, and P. Burlamacchi, IEEE J. Quantum Electronics **QE-2**, 341 [1966].

[17] For further details about this method, see B. L. Morgan and L. Mandel, Phys. Rev. Lett. **16**, 1012 [1966] and the literature cited there.

[18] Special examples of this type have been treated, e. g., by M. N. Rosenbluth and N. Rostoker, Phys. Fluids **5**, 776 [1962], and F. Perkins and E. E. Salpeter, Phys. Rev. **139**, A 55 [1965]. See also Ref [2] and the literature cited there.